Singular and Simplicial Homology Groups

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Abstract

This paper identified the homology groups $H_n = Ker\partial_n/Im\partial_{n+1}$ of r-simplex, n-dimensional simplicial complex and Δ-complex. We show that a 0-simplex (p_0) is a point or a vertex, 1-simplex (p_0p_1) is a line or an edge, a 2-simplex $(p_0p_1p_2)$ is defined to be a triangle with its interior included and a 3-simplex $(p_0p_1p_2p_3)$ is a solid tetrahedron. It is easy to continue to any r-simplex $(p_0p_1 \cdots p_r)$. The aims of this paper are to construct and defined Singular Homology Groups and Simplicial Homology Groups in terms of n-dimensional simplicial complex (K) and a topological space (X). We followed the historical analysis mathematical method. We found that simplexes are building blocks of polyhedron, both approaches yield the same results, and the homology groups (H_n) are quotient groups of r-cycle groups (Z_r) and r-boundary groups (B_r).

Key Words: *Boundary operator, Simplicial complex, Singular homology.*

1. Introduction:

The basic tools such as complexes and incidence numbers for constructing simplicial homology groups where given by Poincare′ in 1985. The basic idea of his construction is that is starts with a geometric object (a space) which is given by combinatorial data (a complex). Then the linear algebra and boundary relations determined by this data are used to construct homology groups. It took more than thirty years to develop homology theory (H_n) applicable to curvilinear polyhedral, embodying the notions given by Poincare′ in 1985. The functor H_n measures the number of ‘n-dimensional’ in the simplicial complex (or in the space), which means that the n-sphere S^n has exactly one n-dimensional hole and there is no m-dimensional hole if $m \neq n$. A 0-dimensional hole is a pair of points in different path components which assert that H_0 measures path connectedness. The simplicial techniques in the simplicial homology theory prescribed by Poincare′ were gradually generalized to singular homology using algebraic properties of the singular complex. The idea of Poincare′ on homology theory was generalized in two directions.

1.From complexes to more general topological spaces, where the homology groups are not characterized by numerical invariants.

2. From the group Z to arbitrary abelian groups [10] p(348).

Homology is the most ingenious invention in algebraic topology. Classically, the definition of homology groups was based on the combinatorial data of simplicial complex. This definition did not yield directly a topological invariant. In learning about homology, one has to follow three lines of thinking at the same time: 1. The construction. 2. Homological algebra. 3. Axiomatic treatment [13] p(223).

In this paper we construct a nontrivial homology theory, singular homology. It is an ordinary homology theory, and the coefficient group in the integer Z. There are two approaches to singular homology: Via singular cubes or singular simplices. Each has advantages and disadvantages, but for our purposes singular cubes are much preferable, and so we use them. (Both approaches yield the same results. The question is which is technically simple) [12] p(55)

2. Simplexes and Singular n-chain Groups:

Definition(2.1): The n-dimensional standard simplex is

$$\Delta^n = \Delta[n] = \{(t_0, \cdots, t_n) \in R^{n+1} | \textstyle\sum_{i=0}^{n} t_i = 1 , t_i \geq 0\} \subset R^{n+1} \qquad (2.1).$$

We set $[n] = \{0, \cdots, n\}$. A weakly increasing map $\alpha: [m] \longrightarrow [n]$ induced an affine map $\Delta[\alpha]: \Delta[m] \longrightarrow \Delta[n], \sum_{i=0}^{m} t_i e_i \longmapsto \sum_{i=0}^{m} t_i e_{\alpha(i)}$.

Here e_i is the standard unit vector, thus $\sum_{i=0}^{m} t_i e_i = (t_0, \cdots, t_m)$. This map satisfy the rule of a functor $\Delta(\alpha \circ \beta) = \Delta(\alpha) \circ \Delta(\beta)$ and $\Delta(id) = id$ [13] p(223).

Definition(2.2): Let $\delta_i^n : [n-1] \longrightarrow [n]$ be the injective map which misses value i then

$$\delta_j^{n+1}\delta_i^n = \delta_i^{n+1}\delta_{j-1}^n, i < j \qquad (2.2).$$

The composition misses i and j. We write $d_i^n = \Delta(\delta_i^n)$ by functoriality, the d_i^n satisfy the analogues commutation rules [13] p(224).

Definition(2.3): A continuous map $\sigma: \Delta^n \longrightarrow X$ is called a singular n-simplex in X. The i^{th} face of σ is $\sigma \circ d_i^n$, we denote by $S_n(X)$ the free abelian group with basis the set of singular n-simplices in X. If $X = \emptyset$, we let $S_n(X) = 0$.

An element $x \in S_n(X)$ is called a singular n-chain. We think of x as a formal finite linear combination $x = \sum_\sigma n_\sigma \sigma, n_\sigma \in Z$ [13] p(224).

Definition(2.4): The boundary operator ∂_q is defined for $q \geq 1$ by

$$\partial_q : S_q(X) \longrightarrow S_{q-1}(X), \sigma \longmapsto \sum_{i=0}^{q} (-1)^i \sigma d_i^q \qquad (2.3)$$

and for $q \leq 0$ as the zero map, $S_q(X)$ called the singular chain groups and the boundary relation is $\partial_{q-1}\partial_q = 0$ [13] p(224).

Definition(2.5): The free abelian group generated by the set of singular q-simlpices of X is called the group of singular q-chains of X and is denoted by $C_q(X)$. Any element of this group is called a singular q-chain of X [9] p(29).

Definition(2.6): The singular chain groups $S_q(X)$ and the boundary operator ∂_q form a chain complex, called the singular chain complex $S.(X)$of X. Its n^{th} homology group is denoted by $H_n(X) = H_n(X; Z)$ and called singular homology group of X (with coefficients in Z) [13] p(224).

Definition(2.7): A continuous map $f: X \longrightarrow Y$ induces a homomorphism

$$f_\#: S_q(f): S_q(X) \longrightarrow S_q(Y), \sigma \longmapsto f\sigma \qquad (2.4).$$

The family of the $S_q(f)$ is a chain map $S_q(f): S.(X) \longrightarrow S.(Y)$. Thus we have induced homomorphism $f_*: H_q(f): H_q(X) \longrightarrow H_q(Y)$[13] p(224).

Theorem(2.8): For a path connected X, $\pi_1(X, x_0) \approx H_1(X)$ where $x_0 \in X$. [5] p(146).

3. Oriented Simplexes:

We may assign orientations to an r-simplex for $r \geq 1$. Instead of $< \cdots >$ for an un oriented simplex, we used $(\cdots)$ to denote an oriented simplex. The symbol σ_r is used to denote both types of simplex.

Definition(3.1): An oriented 1-simplex $\sigma_1 = (p_0p_1)$ is a directed line segment traversed in the direction $p_0 \longrightarrow p_1$. (p_0p_1) should be distinguished from (p_1p_0) we require that

$$(p_0p_1) = -(p_1p_0) \qquad (3.1).$$

Similarly, an oriented 2-simplex $\sigma_2 = (p_0p_1p_2)$ is a triangular region $p_0p_1p_2$ with a prescribed orientation along the edges. Observe that the orientation given by $p_0p_1p_2$ is the same as that given by $p_2p_0p_1$ or $p_1p_2p_0$ but opposite to $p_0p_2p_1$, $p_2p_1p_0$ or $p_1p_0p_2$. We require that

$$(p_0p_1p_2) = (p_2p_0p_1) = (p_1p_2p_0) = -(p_0p_2p_1) = -(p_2p_1p_0) = -(p_1p_0p_2) \quad (3.2).$$

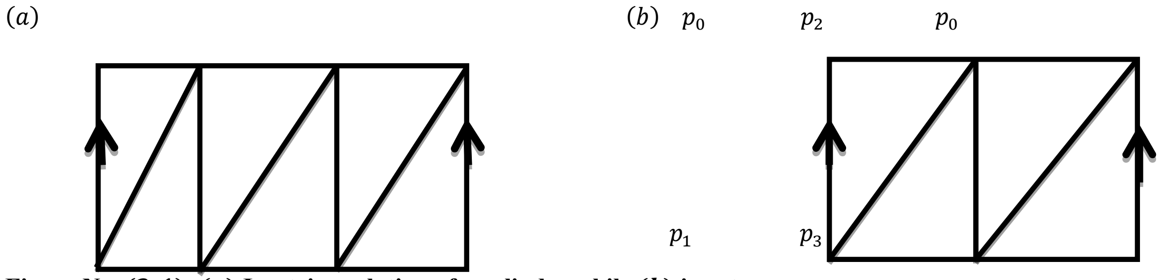

Figure No. (3.1): (*a*) Is a triangulation of a cylinder while (*b*) is not.

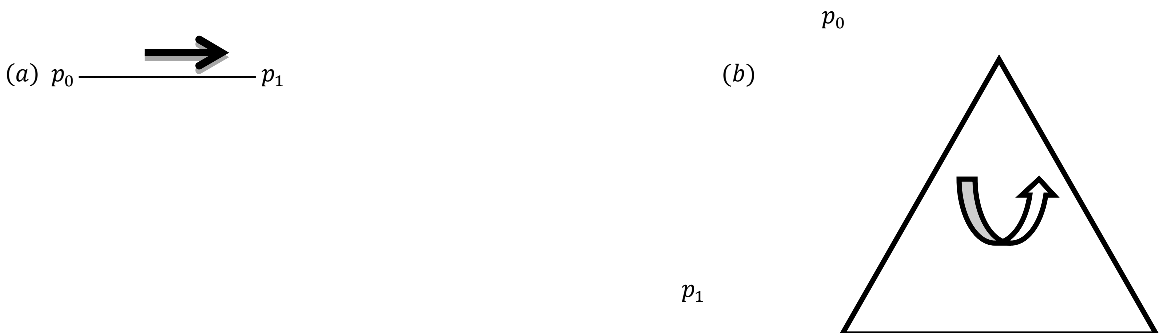

Figure N0. (3.2): An oriented 1-simplex (a) and an oriented 2-simplex (b)

[11] p(120).

Definition(3.2): Let P be a permutation of 0,1,2

$$P = \begin{pmatrix} 0 & 1 & 2 \\ i & j & k \end{pmatrix}$$

These relations are summarized as

$$(p_ip_jp_k) = sgn(P)(p_0p_1p_2) \quad (3.3).$$

Where $sgn(P) = +1(-1)$ if P is an even (odd) permutation [11] p(120).

Definition(3.3): An oriented 3-simplex $\sigma_3 = (p_0p_1p_2p_3)$ is an ordered sequence of four vertices of a tetrahedron. Let

$$P = \begin{pmatrix} 0 & 1 & 2 & 3 \\ i & j & k & l \end{pmatrix}$$

be a permutation. We defined

$$(p_ip_jp_kp_l) = sgn(P)(p_0p_1p_2p_3) \quad (3.4).$$

Definition(3.4): Let $r+1$ geometrically independent points $p_0, p_1, \cdots, p_r$ in R^m. Let $\{p_{i_0}, p_{i_1}, \cdots, p_{i_r}\}$ be a sequence of points obtained by a permutation of the points $p_0, p_1, \cdots, p_r$, we defined $\{p_0, p_1, \cdots, p_r\}$ and $\{p_{i_0}, p_{i_1}, \cdots, p_{i_r}\}$ to be equivalent if

$$P = \begin{pmatrix} 0 & 1 & \cdots & r \\ i_0 & i_1 & \cdots & i_r \end{pmatrix}$$

is an even permutation. Clearly this is an equivalence relation, the equivalence class of which is called an oriented r-simplex. There are two equivalence classes, one consists of even permutations of $p_0, p_1, \cdots, p_r$, the other of odd permutations. The equivalence class (oriented r-simplex) which contains

$\{p_0, p_1, \cdots, p_r\}$ is denoted by $\sigma_r = (p_0 p_1 \cdots p_r)$, while the other is denoted by $-\sigma_r = -(p_0 p_1 \cdots p_r)$. In other words,

$$(p_{i_0}, p_{i_1}, \cdots, p_{i_r}) = sgn(P)(p_0 p_1 \cdots p_r) \quad (3.5).$$

[11] p(121).

Definition(3.5): Let $p_0, p_1, \cdots, p_r$ be points geometrically independent in R^m where $m \geq r$. The r-simplex $\sigma_r = (p_0, p_1, \cdots, p_r)$ is expressed as

$$\sigma^r = \{x \in R^m | x = \sum_{i=0}^{r} c_i p_i, c_i \geq 0, \sum_{i=0}^{r} c_i = 1\} \quad (3.6).$$

$(c_0, c_1, \cdots, c_r)$ is called the barycentric coordinates of x. Since σ_r is a bounded and closed subset of R^m, it is compact. The number of q-faces in an r-simplex is $\binom{r+1}{q+1}$. A 0-simplex is defined to have no proper faces [11] p(118).

4. n^{th} Homology groups of n-dimensional Simplicial Complex:

Let K be a set of finite number of simplexes in R^m. If these simplexes are nicely fitted together, K is called a **simplicial complex**. Let $K = \{\sigma_\alpha\}$ be an n-dimensional simplicial complex. We regard the simplexes σ_α in K as oriented simplexes and denoted them by the same symbol σ_α as remarked before.

Definition(4.1): The r-chain group $C_r(K)$ of a simplicial complex K is a free abelian group generated by the oriented r-simplexes of K. If $r > dimK$, $C_r(K)$ is defined to be 0. An element of $C_r(K)$ is called r-chain [11] p(121).

Definition(4.2): Let there be $I_r r$-simplexes in K. We denoted each of them by $\sigma_{r,i}(1 \leq i \leq I_r)$. Then $c \in C_r(K)$ is expressed by

$$c = \sum_{i=0}^{I_r} c_i \sigma_{r,i}, \qquad c_i \in Z \quad (4.1).$$

The integers c_i are called the coefficients of C. The group structure is given as follows. The addition of two r-chains, $c = \sum_i c_i \sigma_{r,i}$ and $c' = \sum_i c_i' \sigma_{r,i}$ is

$$c + c' = \sum_i (c + c')\sigma_{r,i} \quad (4.2).$$

The unit element is $0 = \sum_i 0 \cdot \sigma_{r,i}$, while the inverse element of c is $-c = \sum_i (-c_i)\sigma_{r,i}$. Thus, $C_r(K)$ is a free abelian group of rank I_r,

$$C_r(K) \cong \underbrace{Z \oplus Z \oplus \cdots \oplus Z}_{I_r} \quad (4.3).$$

Before we define the cycle group and the boundary group, we need to introduce the boundary operator [11] p(122).

Definition(4.3): We now define a homomorphism

$$\partial_r : C_r(K) \longrightarrow C_{r-1}(K) \quad (4.4)$$

∂_r is called the boundary operator. If $\sigma_r = (p_0, p_1, \cdots, p_r)$ is an oriented simplex with $r > 0$, we define

$$\partial_r \sigma_r = \partial_r (p_0, p_1, \cdots, p_r) = \sum_{i=0}^{r} (-1)^r (p_0, p_1, \cdots, \hat{p}_i, \cdots, p_r) \quad (4.5)$$

where the symbol $\hat{p}_i$ mean that the vertex p_i is to be deleted from the array. Since $C_r(K)$ is the trivial group for $r < 0$, the operator ∂_r is the trivial homomorphism for $r \leq 0$ [7] p(38).

Definition(4.4): Let us denoted the boundary of an r-simplex σ_r by $\partial_r\sigma_r$. ∂_r should be understood as an operator acting on σ_r to produce its boundary. Since a 0-simplex has no boundary, we define

$$\partial_0 p_0 = 0 \quad (4.6).$$

For a 1-simplex (p_0p_1), we define

$$\partial_1(p_0p_1) = p_1 - p_0 \quad (4.7).$$

[11] p(122).

Example(4.5):

1. For a 2-simplex, one has $\partial_2(p_0p_1p_2) = (p_1p_2) - (p_0p_2) + (p_0p_1)$.

2. And for a 3-simplex one has the formula $\partial_3(p_0p_1p_2p_3) = (p_1p_2p_3) - (p_0p_2p_3) + (p_0p_1p_3) - (p_0p_1p_2)$ [7] p(39).

Definition(4.6): Let K be an n-dimensional simplicial complex. There exists a sequence of free abelian groups and homomorphisms

$$0 \xrightarrow{i} C_n(K) \xrightarrow{\partial_n} C_{n-1}(K) \xrightarrow{\partial_{n-1}} C_{n-2}(K) \cdots \xrightarrow{\partial_2} C_1(K) \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0 \quad (4.8).$$

Where $i: 0 \hookrightarrow C_n(K)$ is an inclusion map (0 is regarded as the unit element of $C_n(K)$). This sequence is called the chain complex, associated with K and is denoted by $C(K)$. It is interesting to study the image and kernel of the homomorphisms ∂_r [11] p(123).

Definition(4.7): If $c \in C_r(K)$ satisfies $\partial_r c = 0$, c is called an r-cycle. The set of r-cycles $Z_r(K)$ is a subgroup of $C_r(K)$ and is called the r-cycle group. Noted that

$$Z_r(K) = Ker\partial_r \quad (4.9).$$

Definition(4.8): Let K be an n-dimensional simplicial complex and let $c \in C_r(K)$. If there exists an element $d \in C_{r+1}(K)$ such that

$$c = \partial_{r+1} d \quad (4.10).$$

Then c is called an r-boundary. The set of r-boundaries $B_r(K)$ is a subgroup of $C_r(K)$ and is called the r-boundary group. Noted that

$$B_r(K) = Im\partial_{r+1} \quad (4.11).$$

[11] p(124).

Theorem(4.9): Let $Z_r(K)$ and $B_r(K)$ be the r- cycle group and the r-boundary group of $C_r(K)$ then

$$B_r(K) \subset Z_r(K) \qquad (\subset C_r(K)) \quad (4.12).$$

Proof: Any element c of $B_r(K)$ is written as $c = \partial_{r+1} d$ for some $d \in C_{r+1}(K)$. Then we find $\partial_r c = \partial_r(\partial_{r+1} d) = 0$ that is $c \in Z_r(K)$. This implies $Z_r(K) \supset B_r(K)$ [11] p(125).

Definition(4.10): Let K be an n-dimensional simplicial complex, the n^{th} homology group $H_r(K)$, $0 \le r \le n$ associated with K is defined by

$$H_r(K) = Z_r(K)/B_r(K) \quad (4.13).$$

[11] p(125).

Corollary(4.11): If K is a simplicial complex of dimension n, $H_q(K) = 0$ for all $q > 0$ [3] p(175).

Lemma(4.12): The composite map $\partial_r \circ \partial_{r+1}: C_{r+1}(K) \longrightarrow C_{r-1}(K)$ is a zero map, that is, $\partial_r(\partial_{r+1} c) = 0$ for any $c \in C_r(K)$.

Proof: Since ∂_r is a linear operator on $C_r(K)$, it is sufficient to prove the identity $\partial_r \circ \partial_{r+1} = 0$ for the generator of $C_{r+1}(K)$. If $r = 0$, $\partial_0 \circ \partial_1 = 0$, since ∂_0 is a zero operator. Let us assume $r > 0$. Take $\sigma = (p_0, p_1, \cdots, p_r) \in C_{r+1}(K)$. We find

$$\partial_r(\partial_{r+1}\sigma) = \partial_r \sum_{i=0}^{r+1} (-1)^i \,(p_0 \cdots \hat{p}_i \cdots p_{r+1})$$

$$= \sum_{i=0}^{r+1} (-1)^i \,\partial_r (p_0 \cdots \hat{p}_i \cdots p_{r+1})$$

$$= \sum_{i=0}^{r+1} (-1)^i \,(\sum_{j=0}^{i-1} (-1)^j \left(p_0 \cdots \hat{p}_j \cdots \hat{p}_i \cdots p_{r+1}\right)$$

$$+ \sum_{j=i+1}^{r+1} (-1)^{j-1} (\,p_0 \cdots \hat{p}_i \cdots \hat{p}_j \cdots p_{r+1}))$$

$$= \sum_{j<i} (-1)^{i+j} (p_0 \cdots \hat{p}_j \cdots \hat{p}_i \cdots p_{r+1})$$

$$- \sum_{j>i} (-1)^{i+j} \left(p_0 \cdots \hat{p}_i \cdots \hat{p}_j \cdots p_{r+1}\right) = 0 \qquad (4.14).$$

[11] p(124).

Theorem(4.13): For any simplex Δ, $\partial\partial\Delta = 0$ [12] p(2).

Theorem(4.14): The homology groups of a simplicial complex K are finitely generated abelian group [3] p(183).

5. Singular Homology Groups:

Definition(5.1): The standard 0-cube I^0 is the point $0 \in R^0$. For $n \geq 1$, the standard n-cube is

$$I^n = \{x = (x_1, x_2, \cdots, x_n) \in R^n | 0 \leq x_i \leq 1, i = (1,2, \cdots, n)\} \qquad (5.1).$$

It is i^{th} front face is

$$A_i = A_i(I^n) = \{x \in I^n | x_i = 0\} \qquad (5.2)$$

and it is i^{th} back face is

$$B_i = B_i(I^n) = \{x \in I^n | x_i = 1\} \qquad (5.3)$$

[12] p(55).

Definition(5.2): Let I^n be the standard n-cube. It is boundary is given by $\partial I^0 = 0$ and

$$\partial I^n = \sum_{i=1}^{n} (-1)^i (A_i - B_i) \text{ for } n > 0 \qquad (5.4).$$

∂I^n is a free abelian group generated by $\{A_i, B_i | i = 1,2, \cdots, n\}$ [12] p(56).

Lemma(5.3): For any n, $\partial\partial(I^n) = 0$ [12] p(56).

Definition(5.4): Let X be a topological space. A singular n-cube of X is a map $\Phi: I^n \longrightarrow X$. We let $\alpha_i: I^{n-1} \longrightarrow I^n$ be the inclusion of the i^{th} front face and $\beta_i: I^{n-1} \longrightarrow I^n$ be the inclusion of the i^{th} back face, $i.e.$ [35] p(56).

$$\alpha_i(x_1, x_2, \cdots, x_{n-1}) = (x_1, x_2, \cdots, x_{i-1}, 0, x_i, \cdots, x_{n-1})$$

$$\beta_i(x_1, x_2, \cdots, x_{n-1}) = (x_1, x_2, \cdots, x_{i-1}, 1, x_i, \cdots, x_{i-1}) \quad (5.5).$$

Definition(5.5): The chain complex $C(X)$

$$\cdots \longrightarrow C_2(X) \overset{\partial_2}{\to} C_1(X) \overset{\partial_1}{\to} C_0(X) \overset{\partial_0}{\to} 0 \longrightarrow 0 \cdots$$

is the singular chain complex of X [12] p(57).

Definition(5.6): The homology of the singular chain complex of X is the singular homology of X.

$Z_n(X) = Ker(\partial_n: C_n(X) \longrightarrow C_{n-1}(X))$, the group of singular n-cycles,

$B_n(X) = Im(\partial_{n+1}: C_{n+1}(X) \longrightarrow C_n(X))$, the group of singular n-boundaries,

$H_n(X) = Z_n(X)/B_n(X)$, the n^{th} singular homology group of X [12] p(57).

Theorem(5.7): If X is a single point space then $H_0(X)$ is isomorphic with Z and $H_n(X) = 0$ for all $n > 0$ [4] p(100).

Lemma(5.8): If f and g are paths in X such that $f(1) = g(0)$, then the 1-chain $f * g - f - g$ is a boundary [6] p(173).

Lemma(5.9): If f is a path in X then $f + f^{-1}$ is a boundary. Also the constant path is boundary [6] p(173).

Lemma(5.10): $H_k(\sum C^\alpha) \approx \sum_\alpha H_k(C^\alpha)$.

Proof: Noted that by the definition of the chain complex $\sum C^\alpha$ we have

$Z_k(\sum C^\alpha) \approx \sum_\alpha (Z_k(C^\alpha))$ and $B_k(\sum C^\alpha) \approx \sum_\alpha (B_k(C^\alpha))$.

Therefore

$$H_k(\sum_\alpha C^\alpha) = Z_k(\sum C^\alpha)/B_k(\sum C^\alpha)$$

$$= \sum Z_k(C^\alpha)/\sum B_k(C^\alpha)$$

$$= \sum (Z_k(C^\alpha)/B_k(C^\alpha)) = \sum H_k(C^\alpha) \quad (5.6).$$

[8] p(8).

6. Some Properties of Homology Groups:

Definition(6.1): Suppose that X and Y are topological spaces and $f: X \longrightarrow Y$ is continuous map. Consider $H_n(X)$ and $H_n(Y)$ and let $f_*: H_n(X) \longrightarrow H_n(Y)$ be defined by

$$f_*(\sum_{j \in J} n_j \varphi_j) = \sum_{j \in J} n_j f \varphi_j \quad (6.1).$$

Where $\sum_{j \in J} n_j \varphi_j$ is an n-cycle in X. Then f_* is clearly a homomorphism of n^{th} homology groups

$f_*: H_n(X) \longrightarrow H_n(Y)$

This map is called the induced homomorphism. We defined

$f_\#: S_n(X) \longrightarrow S_n(Y)$

by

$$f_{\#}(\sum_{j\in J} n_j\varphi_j) = \sum_{j\in J} n_j f\varphi_j \qquad (6.2).$$

Then $f_{\#}$ is a homomorphism of groups [4] p(102).

Theorem(6.2): $\partial f_{\#} = f_{\#}\partial$ [4] p(102).

Corollary(6.3): $f_{\#}$ sends cycles to cycles and boundaries to boundaries [4] p(102).

Theorem(6.4): Let $X = U_1 \cup U_2$ where U_1 and U_2 are open subsets of X. If is possible to find homomorphisms $\Delta: H_k(X) \longrightarrow H_{k-1}(U_1 \cap U_2)$ with the property that in the following sequence

$$\cdots \longrightarrow H_{k+1}(X) \overset{\Delta}{\to} H_k(U_1 \cap U_2) \overset{i}{\to} H_k(U_1) \oplus H_k(U_2) \overset{j}{\to} H_k(X) \overset{\Delta}{\to} H_{k-1}(U_1 \cap U_2) \quad \longrightarrow \cdots$$

of groups and homomorphisms, the kernel of each homomorphism is equal to the image of the preceding one. The above homomorphisms Δ are called connecting homomorphism [4] p(106).

Theorem(6.5): Let $r: S^n \longrightarrow S^n$ the reflection map defined by $r(x_0, x_1, \cdots x_n) = (-x_0, x_1, \cdots x_n)$. Then the induced homomorphism

$H_n(r): H_n(S^n) \longrightarrow H_n(S^n)$

is multiplication by -1[1] p(94).

Theorem(6.6): If $f: X \longrightarrow Y$ is a homomorphism, then $f_*: H_p(X) \longrightarrow H_p(Y)$ is an isomorphism for each p [8] p(11).

Theorem(6.7): If X is a convex subset of R^n, then $H_p(X) = 0$, for $p > 0$ [4] p(11).

Proposition(6.8): If $f: X \longrightarrow Y$ is a homotopy equivalence, then $f_*: H_n(X) \longrightarrow H_n(Y)$ is an isomorphism for each n [8] p(16).

Corollary(6.9): If $i: A \longrightarrow X$ is the inclusion map of a retract A of X, then $i_*: H_*(A) \longrightarrow H_*(X)$ is a monomorphism into a direct summand. If A is deformation retract of X, then i_* is an isomorphism [8] p(17).

7. Simplicial Homology Groups:

Our goal now is to define the simplicial homology groups of a Δ-complex X. Let $\Delta_n(X)$ be the free abelian group with basis the open n-simplices e_α^n of X. Elements of $\Delta_n(X)$, called n-chains, can be written as finite formal sums $\sum_\alpha n_\alpha e_\alpha^n$ with coefficients $n_\alpha \in Z$ [2] p(104).

Definition(7.1): A Δ-complex structure on a space X is a collection of maps $\sigma_\alpha: \Delta^n \longrightarrow X$, with n depending on the index α, such that

1. The restriction $\sigma_\alpha|\Delta^{0n}$ is injective, and each point of X is in the image of exactly one such restriction $\sigma_\alpha|\Delta^{0n}$.

2. Each restriction of σ_α to a face of Δ^n is one of the maps $\sigma_\beta: \Delta^{n-1} \longrightarrow X$. Here we are identifying the face of Δ^n with Δ^{n-1} by the canonical linear homomorphism between them that preserves the ordinary of the vertices.

3. A set $A \subset X$ is open if and only if $\sigma^{-1}(A)$ is open in Δ^n for each σ_α [2] p (103).

So we can define the n^{th} homology group of the chain complex to be the quotient group $H_n = Ker\partial_n / Im\partial_{n+1}$. Elements of $Ker\partial_n$ are called cycles and elements of $Im\partial_{n+1}$ are boundaries. Elements of H_n are cosets of $Im\partial_{n+1}$ called homology classes. Two cycles representing the same homology class are said to be homologous. This mean their difference is a boundary. Returning to the case that $C_n = \Delta_n(X)$, the homology group $Ker\partial_n / Im\partial_{n+1}$ will be denoted $H_n^\Delta(X)$ and called the n^{th} simplicial homology group of X [2] p(106).

Example(7.2): $X = S^1$, with one vertex v and one edge e. Then $\Delta_0(S^1)$ and $\Delta_1(S^1)$ are both Z and the boundary map ∂_1 is zero, since $\partial e = v - v$. The group $\Delta_n(S^1)$ are zero for $n \geq 2$, since there are no simplices in these dimensions. Hence

$$H_n^{\Delta}(S^1) \approx \begin{cases} Z & for\ n = 0,1 \\ 0 & for\ n \geq 2 \end{cases}$$

Example(7.3): $X = T$, the torus with the Δ-complex structure pictured earlier, having one vertex, three edges a, b and c, and two 2-simplices U and L. As in the previous example, $\partial_1 = 0$ so $H_0^{\Delta}(T) \sim Z$ and $\{a, b, a + b - c\}$ is a basis for $\Delta_1(T)$, it follows that $H_1^{\Delta}(T) \sim Z \oplus Z$ with basis the homology class $[a]$ and $[b]$. Since there are no 3-simplices, $H_2^{\Delta}(T)$ is equal to $Ker\partial_2$, which is infinite cyclic generated by $U - L$, since $\partial(pU + qL) = p + q(a + b - c) = 0$ only if $p = -q$. Thus

$$H_n^{\Delta}(T) \approx \begin{cases} Z \oplus Z & for\ n = 1 \\ Z & for\ n = 0,2 \\ 0 & for\ n \geq 3 \end{cases}$$

[2] p(106).

Corollary(7.4): There exists no continuous map $r: D^n \longrightarrow S^{n-1} = \partial D^n$ such that $r(x) = x$ for all $x \in S^{n-1}$ [14] p(11).

Proposition(7.5): Corresponding to the decomposition of a space X into its path components X_α there is an isomorphism of $H_n(X)$ with the direct sum $\oplus_\alpha H_n(X_\alpha)$.

Proof: Since a singular simplex always has path-connected image , $C_n(X)$ splits as the direct sum of subgroups $C_n(X_\alpha)$. The boundary map ∂_n preserve this direct sum decomposition, taking $C_n(X_\alpha)$ to $C_{n-1}(X_\alpha)$, so $Ker\partial_n$ and $Im\partial_{n+1}$ split similarly as direct sums, hence the homology groups also split

$$H_n(X) \approx \oplus_\alpha H_n(X_\alpha) \quad (7.1).$$

[2] p(109).

Proposition(7.6): If F and G are homotopic maps from $X$to Y, then $F_* = G_*$ [15] p(93).

Corollary(7.7): If X is path connected, then $\chi: \pi_1(X, x_0) \longrightarrow H_1(X; Z)$ is an isomorphism if and only if the fundamental group of X is commutative [1] p(63).

Proposition(7.8): A chain map between chain complexes induce homomorphisms between the homology groups of the two complexes [2] p(111).

Results:

The n^{th} homology groups of the chain complex and chain simplex is the quotient group $H_n = Ker\partial_n / Im\partial_{n+1}$. Elements of H_n are cosets of $Im\partial_{n+1}$, singular homology groups and simplicial homology groups yield the same results, an element $c \in C_n(X)$ can be written as a finite of forms $\sum_\alpha n_\alpha e_\alpha^n$, and if $f: X \longrightarrow Y$ is continuous then $f_*: H_n(X) \longrightarrow H_n(Y)$ is a homomorphism of homology groups.

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